Quantum aspects of a noncommutative supersymmetric kink

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ABSTRACT: We consider quantum corrections to a kink of noncommutative supersymmetric φ^4 theory in $1 + 1$ dimensions. Despite the presence of an infinite number of time derivatives in the action, we are able to define supercharges and a Hamiltonian by using an unconventional canonical formalism. We calculate the quantum energy E of the kink (defined as a half-sum of the eigenfrequencies of fluctuations) which coincides with its' value in corresponding commutative theory independently of the noncommutativity parameter. The renormalization also proceeds precisely as in the commutative case. The vacuum expectation value of the new Hamiltonian is also calculated and appears to be consistent with the value of the quantum energy E of the kink.

KEYWORDS: Solitons Monopoles and Instantons, Non-Commutative Geometry.

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Contents

1. Introduction

The study of quantum corrections to solitons in $1 + 1$ dimensions started in 1970's $[1-4]$, and since that time a considerable progress has been made (see [5] for a recent review). Noncommutative (NC) solitons [6, 7] were included in these studies only recently [8, 9]. The work [8] used the small θ expansion, while the paper [9] was concentrated on moderate and large values of the NC parameter. Both papers left many questions unanswered, mostly related to the renormalization and to the possibility of a smooth extension of the results to the region of large (respectively, small) noncommutativity. Besides, in $1+1$ dimensions one deals with time-space noncommutativity which brings an infinite number of time derivatives into the action, so that the very definition of energy becomes less obvious.

Another line of research considers quantum corrections to supersymmetric solitons [10, 11]. It was found [12], that naive arguments leading to zero quantum corrections to the mass of supersymmetric solitons were incorrect, and a new anomaly (the anomaly in the central charge $[13-15]$) was discovered. Taking this anomaly into account restores saturation of the BPS bound at the quantum level.

In this paper we consider quantum correction to the mass of an NC supersymmetric kink in $1 + 1$ dimensions. Our motivation is twofold. First, it is interesting to study the interplay between supersymmetry and noncommutativity with this particular example. Second, supersymmetry simplifies the structure of divergences of quantum field theory and may help to resolve some problems existing in the non-supersymmetric case. Practically, we adapt the methods developed earlier in [16] to the NC case. Supersymmetrization of the NC space-time is done in the most straightforward way $[17-19]$ where only the bosonic

coordinates are deformed. The model we study in here is a supersymmetric extension on the NC φ^4 model in $1 + 1$ dimensions.

In time-space NC theories there are well-known difficulties with the construction of a canonical formalism (due to the presence of an infinite number of time derivatives). Besides, generically there are no locally conserved currents corresponding to global classical symmetries. Therefore, it is a priori unclear whether one can define supercharges in such theories. However, as we show below, this task can be successfully addressed in the framework of an unconventional canonical formalism [20], so that one can introduce supercharges whose brackets give an analog of the Hamiltonian and a central charge. The Hamiltonian has the meaning of the energy integrated over an interval T of time. For a static field configuration it simply reads TE, where E is the energy. The main reason to call these quantities supercharges and a Hamiltonian is that with respect to the new brackets they indeed generate global supesymmetry transformations and the equations of motion, respectively.

Static solutions in NC models in $(1 + 1)$ dimensions are not deformed, i.e. they are the same as in corresponding commutative models. The equations of motion for small fluctuations above such solutions are deformed, and the fluctuations are described by wave equations with frequency-dependent potentials. Nevertheless, in the model we consider, bosonic and fermionic modes are isospectral. To use all advantages of the isospectrality, we employ the zeta-function regularization and make the spectrum discrete by introducing boundaries in the spatial direction. (These boundaries are removed at the end of the calculations). The quantum energy is defined as one half the sum over the eigenfrequencies. The width of the effective potential in the wave equations for the fluctuations with the frequency ω is proportional to $\theta\omega$, where θ is the NC parameter. To keep boundaries far away from the location of the potential we have to make the position of the boundaries frequency-dependent [9]. In this approach, quantum energy of the kink is defined as the energy of a system consisting of the kink and the boundaries minus the (Casimir) energy of the boundaries [16]. For the renormalization, we use the heat kernel subtraction scheme which was shown to be equivalent to the no-tadpole condition in the commutative case [16]. The divergences are removed by a renormalization of the mass, which is precisely the same as in the commutative case. The renormalized energy (mass shift of the kink) does not depend on θ and coincides with its' commutative value.

Keeping in mind future applications to the verification of the quantum BPS bound saturation, we also calculate quantum corrections to the new Hamiltonian. We find the value TE, where E is the mass shift of the soliton. Two apparently different definitions of the quantum energy give consistent results. Also, the renormalization required for the Hamiltonian is the same mass renormalization which we described above.

This paper is organized as follows. In the next section we introduce a classical action and collect some preliminary information. In section 3 we study the new unconventional definition of the canonical algebra, and define supercharge, the Hamiltonian, and the central charge. In section 4 we study the spectrum of fluctuations above the kink. Quantum corrections to the mass of the kink are calculated in section 5, and corrections to the new Hamiltonian are considered in section 6. Concluding remarks are given in section 7.

2. The classical action

We shall describe noncommutativity of the space-time coordinates by the Moyal product

$$(f \star g)(x) = \left[\exp\left(\frac{i}{2} \Theta^{\mu\nu} \partial_\mu^x \partial_\nu^y \right) f(x)g(y) \right]_{y^\mu = x^\mu}, \qquad (2.1)$$

where $\Theta^{\mu\nu}$ is a constant skew-symmetric matrix which can be chosen as $\Theta^{\mu\nu} = 2\theta\epsilon^{\mu\nu}$ with $\epsilon^{01} = 1$. After splitting the coordinates into time and space, $\{x^\mu\} = \{t, x\}$, we have the following useful formulae

$$f(x) \star e^{i\omega t} = e^{i\omega t} f(x + \theta\omega), \quad e^{i\omega t} \star f(x) = e^{i\omega t} f(x - \theta\omega). \qquad (2.2)$$

The Moyal product is closed,

$$\int d^2x \, f_1 \star f_2 = \int d^2x \, f_1 \cdot f_2, \qquad (2.3)$$

and has the property that

$$\int d^2x \, f_1 \star f_2 = (-1)^{g_1 g_2} \int d^2x \, f_2 \star f_1, \qquad (2.4)$$

where the grading $g_i = 0$ if f_i is bosonic, and $g_i = 1$ if f_i is fermionic. To derive the properties (2.3) and (2.4) one has to integrate by parts in (2.1). In general, boundary terms may appear. To avoid them, we assume that in the time direction all fields are periodic with a very large period which should be sent to infinity at the end. In the spatial directions all fields must approach constant values sufficiently fast. Such boundary conditions are satisfied by static solitons and classical variations of the fields which produce the equations of motion. A different set of boundary conditions will be used in section 4 to analyze quantum fluctuations.

The action for a supersymmetric NC φ^4 model reads

$$S = -\frac{1}{2} \int_\mathcal{M} d^2x \, \left((\partial_\mu \varphi)^2 + U'(\varphi) \star \bar{\psi} \star \psi + \bar{\psi}\gamma^\mu \partial_\mu \psi - 2F \star U - F^2 \right). \qquad (2.5)$$

Here φ is a real scalar field, and ψ is a Majorana spinor. We take γ-matrices in the Majorana representation

$$\gamma^0 = -i\sigma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (2.6)$$

In this representation the components of ψ are real. $\bar{\psi} = \psi^T i\gamma^0$. Components of the spinors will be marked by the subscripts $\pm$, so that $\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$, $\epsilon = \begin{pmatrix} \epsilon_+ \\ \epsilon_- \end{pmatrix}$. For the φ^4 model

$$U(\varphi) = \sqrt{\frac{\lambda}{2}}(v_0^2 - \varphi \star \varphi), \qquad U'(\varphi) = -\sqrt{2\lambda}\varphi. \qquad (2.7)$$

Note, that though due to (2.3) one star can always be deleted under an integral, it is more convenient to write all stars explicitly in all terms higher that second order in the fields since mixed (star with ordinary) products are not associative.

The supersymmetry transformations

$$\delta\varphi = \bar{\epsilon}\psi, \qquad \delta\psi = (\gamma^\mu\partial_\mu\varphi + F)\epsilon, \qquad \delta F = \bar{\epsilon}\gamma^\mu\partial_\mu\psi. \tag{2.8}$$

are linear, and, therefore, are undeformed. The invariance of (2.5) under (2.8) follows from the general analysis of [18, 19], but can also be verified directly.

The auxiliary field F may be excluded by means of its' algebraic[1] equation of motion

$$F = -U(\varphi). \tag{2.9}$$

The action (2.5) becomes

$$S = -\frac{1}{2}\int_{\mathcal{M}} d^2x \left((\partial_\mu\varphi)^2 + U'(\varphi) \star \bar{\psi} \star \psi + \bar{\psi}\gamma^\mu\partial_\mu\psi + U \star U\right), \tag{2.10}$$

and the supersymmetry transformations read

$$\delta\varphi = \bar{\epsilon}\psi, \qquad \delta\psi = (\gamma^\mu\partial_\mu\varphi - U)\epsilon. \tag{2.11}$$

The equations of motion corresponding to the action (2.10) are

$$\partial_\mu\partial^\mu\varphi + \sqrt{\frac{\lambda}{2}}\,\bar{\psi} \star \psi - \frac{1}{2}(U \star U' + U' \star U) = 0, \tag{2.12}$$

$$\slashed{\partial}\psi + \frac{1}{2}(U' \star \psi + \psi \star U') = 0. \tag{2.13}$$

Static solutions of these equations are the same as in the commutative case. In particular, there is the kink solution

$$\Phi(x) = v_0 \tanh\left(v_0\sqrt{\frac{\lambda}{2}}\,x\right). \tag{2.14}$$

This solution satisfies the Bogomolny equation

$$\partial_1\Phi(x) = U(\Phi) \tag{2.15}$$

and is invariant under the supersymmetry transformations (2.11) with $\epsilon_- = 0$.

3. Canonical realization of the supersymmetry algebra

We have no locally conserved supercurrent in the model (as there is no locally conserved energy-momentum tensor in NC theories [21]), but still by using an unconventional canonical formalism for time-space noncommutative theories [20] we can define supercharges which generate the supersymmetry transformations (2.11). Let us briefly outline the formalism of [20] (in [20] only the bosonic case was considered, but an extension to the

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[1]This means that no derivatives acting on F appear.

presence of fermions is straightforward). The canonical pairs are defined ignoring the time derivatives hidden in the star-product. In our model this implies that they are precisely the same as in the commutative case. To read off the symplectic form, let us re-write the action (2.10) in a "hamiltonian" form

$$
\begin{aligned}
S &= \int d^2x \left(-\frac{i}{2}(\partial_0\psi_+ \cdot \psi_+ + \partial_0\psi_- \cdot \psi_-) + \frac{1}{2}((\partial_0\varphi)p - (\partial_0 p)\varphi) - \mathcal{H} \right) \qquad (3.1) \\
&= \int d^2x \left(-\frac{1}{2}(C^{-1})^{AB}\partial_0 z_A \cdot z_B - \mathcal{H} \right).
\end{aligned}
$$

(We use the conventions of Henneaux [22]). Here $\{z_A\} \equiv \{\varphi, p, \psi_+, \psi_-\}$.

$$
\mathcal{H} = \frac{1}{2}\left((\partial_1\varphi)^2 + p^2 + U \star U + U' \star \bar\psi \star \psi + \bar\psi\gamma^1\partial_1\psi\right) \qquad (3.2)
$$

does not contain explicit time derivatives (all time derivatives are hidden in the star product).

The canonical brackets are taken between variables at different times and are postulated to be proportional to *two-dimensional* delta-functions instead of one-dimensional ones, $\{z_A(t,x), z_B(t',x')\} = C_{AB}\delta(t-t')\delta(x-x')$. More explicitly,

$$
\{\varphi(t,x), p(t',x')\} = \delta(x-x')\delta(t-t'), \qquad (3.3)
$$
$$
\{\psi_\pm(t,x), \psi_\pm(t',x')\} = -i\delta(x-x')\delta(t-t'), \qquad (3.4)
$$

and $p = \partial_0\varphi$. Usual grading rules are understood. Now we have to extend the definition of the brackets to star-polynomials of z_A and their derivatives. Here we face a difficulty since a star product by a delta-function is not a well defined object. However, we can define brackets between space-time integral of polynomials. Let F, G be two such integrals. Then

$$
\{F,G\} = \int d^2x \frac{\delta^r F}{\delta z_A(x)} \star C_{AB} \frac{\delta^l G}{\delta z_B(x)}. \qquad (3.5)
$$

Here δ^r and δ^l are right and left variational derivatives. For a practical use, the formula (3.5) has to be understood in the following way. One has to take all pairs of canonical variables z_A, z_B in F and G respectively, then one uses the property (2.4) to bring z_A to the rightmost position in F, and z_B to the leftmost position in G. Then one integrates by parts to remove all explicit derivatives form z_A and z_A. Then one deletes z_A and z_B, star-multiply the expressions obtained, contracts with C_{AB} and integrates over the space-time. The brackets defined in this way satisfy the (graded) Jacobi identities. For bosonic theories this was demonstrated in [20], and an extension to fermions is straightforward.

By taking $F = \int f \star \hat F$, where f is a smooth function, calculating the bracket with G, and then varying with respect to f, one can extend the definition to brackets between star-polynomials $\hat F$ and integrated star-polynomials G. This trick does not work twice. Therefore, it is not possible to define a bracket between unintegrated polynomials, but we shall not need such an object.

In [20] it was shown that these unconventional Poisson brackets can be used to define first-class constraints and generate gauge transformations in time-space NC theories (see

also [23] for an example of practical use of these brackets). Here we shall apply them to analyze global symmetries.

First we note, that if we define the "Hamiltonian" as a space-time integral

$$H = \int d^2x\, \mathcal{H} \tag{3.6}$$

of the density (3.2), then the brackets with H generate the equations of motion

$$\{H, z_A\} = -\partial_0 z_A\,. \tag{3.7}$$

A definition of the "supercharge" then follows by an educated guess as a suitable generalization of corresponding commutative expression. Let us take

$$Q = -\int d^2x (\slashed{\partial}\varphi + U(\varphi)) \star \gamma^0 \psi\,. \tag{3.8}$$

It is easy to check that this "supercharge" indeed generates the supesymmetry transformations

$$\{\bar{\epsilon}Q, z_A\} = -\tilde{\delta} z_A \tag{3.9}$$

of the Hamiltonian action (3.1). On shell the transformations $\tilde{\delta}$ coincide with (2.11).

We see, that the "Hamiltonian" and the "supercharge" possess the characteristic features which we expect from a Hamiltonian and a supercharge. Therefore, we shall sometimes omit the quotation marks in what follows.

The kink solution (2.14) is invariant under the ϵ_+ transformations, which are generated by Q_-. The bracket of two such supercharges reads

$$\{Q_-, Q_-\} = -2i(H - Z) \tag{3.10}$$

where[2]

$$Z = \int d^2x\, \partial_1 W(\varphi), \tag{3.11}$$

$$W(\varphi) = \sqrt{\frac{\lambda}{2}} \left(v_0^2 \varphi - \frac{1}{3}\varphi \star \varphi \star \varphi \right). \tag{3.12}$$

Z is a natural generalization of the central charge to the NC case. We obtained a standard form of a central extension of the supersymmetry algebra in a topologically non-trivial sector [24], though the generators are given by two-dimensional integrals and the brackets are unconventional.

On the kink background both H and Z are divergent unless one restricts the integration over t to a finite interval. Note, that the difference $H - Z$ for the kink is finite and vanishes.

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[2]Note, that there is another total derivative term in $\{Q_-, Q_-\}$, namely $-i\int \partial_1(\bar{\psi}\psi)$. This term vanishes if one considers fluctuations above the kink solution with the asymptotic conditions we discussed above. However, such terms are important for the "supersymmetry without boundary conditions" approach [25].

4. Fluctuations

The spectrum of fluctuations is defined by the linearized equations of motion (2.12) and (2.13). For the fermionic fluctuations we have

$$\begin{pmatrix} \partial_1 + \frac{1}{2}(L(U'(\Phi)) + R(U'(\Phi))) & -\partial_0 \\ \partial_0 & -\partial_1 + \frac{1}{2}(L(U'(\Phi)) + R(U'(\Phi))) \end{pmatrix} \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} = 0. \quad (4.1)$$

Here L and R denote left and right Moyal multiplications respectively, $f_1 \star f_2 = L(f_1)f_2 = R(f_2)f_1$. The fluctuation operator commutes with ∂_0. Consequently, we can look for the solutions in the form

$$\psi_\pm(t, x) = e^{i\omega_f t}\psi_\pm(\omega_f, x). \quad (4.2)$$

The equation (4.1) then yields

$$i\omega_f \psi_+(\omega_f, x) = \left(\partial_1 - \frac{1}{2}(U'(\Phi_+) + U'(\Phi_-)) \right)\psi_-(\omega_f, x),$$

$$i\omega_f \psi_-(\omega_f, x) = \left(\partial_1 + \frac{1}{2}(U'(\Phi_+) + U'(\Phi_-)) \right)\psi_+(\omega_f, x), \quad (4.3)$$

where

$$\Phi_\pm(x) \equiv \Phi(x \pm \theta\omega). \quad (4.4)$$

The property (2.2) of the Moyal product has been used. By iterating the equations (4.3) one obtains

$$\omega_f^2 \psi_+(\omega_f, x) = -D_-(\omega_f)D_+(\omega_f)\psi_+(\omega_f, x),$$

$$\omega_f^2 \psi_-(\omega_f, x) = -D_+(\omega_f)D_-(\omega_f)\psi_-(\omega_f, x), \quad (4.5)$$

where

$$D_\pm(\omega) = \partial_1 \mp \sqrt{\frac{\lambda}{2}}(\Phi_+ + \Phi_-). \quad (4.6)$$

In the bosonic sector, we decompose the scalar field as $\varphi = \Phi + \phi$. The fluctuations ϕ satisfy the linearized field equation

$$-\partial_0^2 \phi = -(\partial_1^2 + \lambda v_0^2 - \lambda(L(\Phi^2) + R(\Phi^2) + L(\Phi)R(\Phi)))\phi. \quad (4.7)$$

Again, we look for the solutions in the form $\phi(\omega_b, x) = e^{i\omega t}\phi(\omega_b, x)$. The equation (4.7) yields

$$\omega_b^2 \phi(\omega_b) = -(\partial_1^2 + \lambda v_0^2 - \lambda(\Phi_+^2 + \Phi_-^2 + \Phi_+\Phi_-))\phi(\omega_b). \quad (4.8)$$

By using the Bogomolny equation (2.15) we obtain

$$\omega_b^2 \phi(\omega_b) = -D_+(\omega_b)D_-(\omega_b)\phi(\omega_b). \quad (4.9)$$

The spectrum of the eigenfrequencies is defined by two operators, $P_1(\omega) = -D_+(\omega)D_-(\omega)$ and $P_2(\omega) = -D_-(\omega)D_+(\omega)$. Due to the intertwining relations

$$P_1(\omega)D_+(\omega) = D_+(\omega)P_2(\omega), \qquad D_-(\omega)P_1(\omega) = P_2(\omega)D_-(\omega) \quad (4.10)$$

these operators are isospectral up to zero modes. Indeed, these relations imply that if $P_1\psi_1 = \lambda\psi_1$, then $D_-\psi_1$ is an eigenfunction of P_2 with the same eigenvalue. Also, if $P_2\psi_2 = \lambda\psi_2$, then $P_1(D_+\psi_2) = \lambda(D_+\psi_2)$.

An explicit form of P_1 follows from (4.8). For the sake of completeness we also present

$$P_2(\omega) = -(\partial_1^2 - \lambda v_0^2 - \lambda\Phi_+\Phi_-). \tag{4.11}$$

In calculations of the quantum corrections it is convenient to go from the continuous to discrete spectrum of P_1 and P_2 by introducing boundaries [16] in the x-direction. We like the boundary to interact with the soliton as weak as possible. Therefore, the boundary should be far away from the place where the kink is localized. However, as we see e.g. from eq. (4.8), the width of the effective potential is proportional to $\theta\omega$ and becomes infinite for $\omega \to \infty$. No boundary seems to be sufficiently far away. To overcome this difficulty, in [9] it was suggested to make the boundary ω-dependent, i.e. to place it to the points $x = \pm l(\omega) = \pm(l_0 + \theta\omega)$ with a large l_0. Having a boundary, one has to impose some boundary conditions on the fluctuations. Particular choice of the boundary conditions is not too important (as anyway we are going to subtract the vacuum energy related to the boundary), but too use the full strength of supersymmetry it is convenient to take supersymmetric boundary conditions which respect the intertwining relations (4.10) and, therefore, preserve isospectrality of $P_1(\omega)$ and $P_2(\omega)$ for any ω. The simplest choice is to impose the Dirichlet boundary conditions on ϕ and ψ_-,

$$\phi|_{x=\pm l(\omega)} = \psi_-|_{x=\pm l(\omega)} = 0. \tag{4.12}$$

The intertwining relations then require a Robin (generalized Neumann) boundary condition for ψ_+:

$$D_+\psi_+|_{x=\pm l(\omega)} = 0. \tag{4.13}$$

(Note that the same boundary condition on ψ_+ follows from the consistency of the Dirac equation (4.3)).

In general, the Moyal product cannot be restricted to an interval with frequency dependent boundaries. However, for operators commuting with the time derivatives (in particular, for Moyal multiplications by a time-independent function) such a restrictions can be made along the lines described in this section.

5. Quantum corrections to the mass

Here we use a generalization of the method [16] to the NC case. Namely, we first consider the kink with boundaries with fluctuations subject to the boundary conditions (4.12) and (4.13), calculate the total quantum energy of this system E_{k+b}, and then subtract the vacuum energy E_b which is due to the presence of boundaries. The energy associated with the kink is then

$$E_{\mathbf{k}} = E_{k+b} - E_b. \tag{5.1}$$

The vacuum energy for each of the systems is defined as a half-sum of the eigenfrequencies,

$$E = \frac{1}{2} \sum \omega_b - \frac{1}{2} \sum \omega_f \qquad (5.2)$$

(we set $\hbar = 1$). In time-space NC theories there is no standard canonical Hamiltonian to justify this formula for the energy (though, there is a non-standard one, see section 3 and 6). For systems with a finite number of additional time derivatives (with fields in stationary but non-static geometries being an example of such systems) it was shown that this definition of the energy is equivalent to the canonical one, and the presence of extra time derivative (which results in modifications of the Klein-Gordon current and corresponding scalar product) influences the results of quantum computations through modification of the spectral density [26, 27] (see also [28, 9] for an extension of this analysis to NC case). Adopting the same approach here looks as the most reliable extension of the notion of vacuum energy to time-space NC theories.

Because of the presence of boundaries we deal with a discrete spectrum of eigenfrequencies. It is convenient to use the zeta-function regularization [29, 30]. The operator P_1 (resp., P_2) is a product of a first-order operator and its' formal adjoint. Therefore, both P_1 and P_2 are non-negative. In the positive spectrum, the zeta-regularized energy reads

$$E(s) = \frac{\mu^{2s}}{2} \left({\sum}'(\omega_b^2)^{\frac{1}{2}-s} - {\sum}'(\omega_f^2)^{\frac{1}{2}-s} \right), \qquad (5.3)$$

where prime tells us that the summation runs over the positive spectrum only. (Zero frequencies do not contribute to the vacuum energy). The parameter μ of the dimension of the mass is introduced in order to keep right dimensionality of the energy independently of the regularization parameter s. Both sums on the right hand side of (5.3) are convergent for $\mathrm{Re}\,(s)$ sufficiently large. At the end of the calculations the result must be analytically continued to the physical value $s = 0$.

Let us first analyze $E_{\mathrm{k+b}}$. Due to the isospectrality properties discussed above

$$E(s)_{\mathrm{k+b}} = 0, \qquad (5.4)$$

i.e., the regularized vacuum energy vanished identically.

Although, obviously, the vacuum energy (5.4) is not divergent, there might be some finite contribution due to a finite renormalization.[3] To define such a contribution one should fix a normalization condition or a subtraction scheme. Here we use the heat-kernel subtraction scheme which is frequently employed in the Casimir energy calculations and is discussed in detail in [33, 34]. Consider a (bosonic) system in $1 + 1$ dimensions with a discrete frequency spectrum $\{\omega_n\}$. Let $k_n^2 = \omega_n^2 - m^2$, where m is the mass (or, the asymptotic value of the potential). The regularized vacuum energy for this system admits a representation,

$$\frac{\mu^2}{2} \sum_n (k_n^2 + m^2)^{\frac{1}{2}-s} = \frac{\mu^2}{2} \int_0^\infty \frac{d\tau}{\tau} \frac{\tau^{s-\frac{1}{2}}}{\Gamma\left(s - \frac{1}{2}\right)} K(\tau) e^{-\tau m^2}, \qquad (5.5)$$

---

[3]This indeed happens in some models. For example, the whole correction to the mass of the supersymmetric Abrikosov-Nielsen-Olesen vortex is due to a finite renormalization of couplings [31, 32].

where

$$K(\tau) = \sum_n e^{-\tau k_n^2} \tag{5.6}$$

is the corresponding heat kernel. Usually, the heat kernel admits an asymptotic expansion[4]

$$K(\tau) \simeq \sum_{p>0} a_p \tau^{p-1} \tag{5.7}$$

as $\tau \to +0$. For $s = 0$ contributions to (5.5) from the terms with $p = 0, 1, 2$ are divergent at the lower limit. We define the divergent part of the vacuum energy as

$$
\begin{aligned}
E^{\mathrm{div}} &\equiv \frac{\mu^2}{2} \int_0^\infty \frac{d\tau}{\tau} \frac{\tau^{s-\frac{1}{2}}}{\Gamma\left(s - \frac{1}{2}\right)} \sum_{n=0}^{2} a_n \tau^{n-1} e^{-\tau m^2} \\
&= \frac{\mu^2}{2\Gamma\left(s - \frac{1}{2}\right)} \left\{ a_0 \Gamma(s-1) m^{2-2s} + a_1 \Gamma\left(s - \frac{1}{2}\right) m^{1-2s} + a_2 \Gamma(s) m^{-2s} \right\}.
\end{aligned} \tag{5.8}
$$

The renormalized energy is then

$$E^{\mathrm{ren}} = [E(s) - E^{\mathrm{div}}(s)]_{s=0} \tag{5.9}$$

This subtraction scheme has two important advantages. First, in the case of commutative scalar theories in $1 + 1$ dimensions it is equivalent [16] to the "no tadpole" normalization condition which is commonly used to calculate the mass shift of two-dimensional solitons. Second, this scheme can easily be extended to the NC case.

Let us return to $E_{\mathrm{k+b}}$. Due to (5.4) the heat kernel is also identically zero, as well as all heat kernel coefficients and $E_{\mathrm{k+b}}^{\mathrm{div}}(s)$. We conclude, that

$$E_{\mathrm{k+b}}^{\mathrm{ren}} = 0. \tag{5.10}$$

Next we have to study the vacuum energy E_{b} due to the presence of boundaries. Far away from the kink, the excitations are free bosonic and fermionic modes with the mass $m = v_0 \sqrt{2\lambda}$ which is defined by asymptotic values of the potential in (4.8) and (4.11). In the bosonic sector, the boundary conditions are Dirichlet. In the fermionic sector, one mode satisfies the Dirichlet conditions as well, another one satisfies the Robin boundary conditions (each of the modes carries one half of a degree of freedom).[5]

Let us study the Robin sector first. For large l_0, (we remind that $l(\omega) = l_0 + \theta\omega$) the condition (4.13) yields

$$(\partial_x + S_1)\psi|_{x=-l(\omega)} = 0, \qquad (-\partial_x + S_2)\psi|_{x=l(\omega)} = 0, \tag{5.11}$$

---

[4]Such an expansion indeed exists for practically all case appearing in the context of quantum field theory. A more precise and complete information on the heat kernel expansion can be found in [37] for commutative space, and in [38] in the NC case. The heat kernel for frequency-dependent problems was analyzed in [27, 35, 36].

[5]It is important that, as in the commutative case [16], we use for the fermions an asymptotic form of the squared Dirac equation (4.5). One cannot substitute asymptotic values of the fields in the Dirac equation (4.1) itself and then extend it smoothly to the whole space $[-l, l]$.

where

$$S_1 = S_2 = v_0\sqrt{2\lambda} \equiv S. \tag{5.12}$$

There are no bound states ($\omega^2 < m^2$) for these boundary conditions. The spectrum of oscillating modes, $\psi = A\sin(kx) + B\cos(kx)$, $k = \sqrt{\omega^2 - m^2}$ is given by solutions of the equation [16]

$$0 = f(\alpha_1, \alpha_2; k) \equiv \sin(2kl(\omega) + \alpha_1 + \alpha_2) \tag{5.13}$$

with

$$\alpha_{1,2} = -\arctan(k/S_{1,2}) \equiv \alpha. \tag{5.14}$$

It is easy to see, that the spectrum in the Dirichlet sector is defined by the equation

$$0 = f(0,0;k). \tag{5.15}$$

Next we represent the vacuum energy as a contour integral [39, 16]. The function $\partial_k \ln f(k)$ has poles with unit residues at the points where $f(k) = 0$. Therefore, we can write

$$E_{\rm b}(s) = -\frac{\mu^{2s}}{4} \oint \frac{dk}{2\pi i}(k^2 + m^2)^{\frac{1}{2}-s}\frac{\partial}{\partial k}(\ln f(\alpha, \alpha; k) - \ln f(0,0;k)), \tag{5.16}$$

where the contour goes anticlockwise around the positive real semiaxis. Along the upper part of the contour we approximate $\sin(2(kl(\omega) + \alpha))$ by $-(1/2i)\exp(-2i(kl(\omega) + \alpha))$ since the term $\exp(2i(kl(\omega) + \alpha))$ vanishes as $l_0 \to \infty$. Along the lower part we keep $(1/2i)\exp(2i(kl(\omega) + \alpha))$. Then,

$$E_{\rm b}(s) = -\mu^{2s}\int_0^\infty \frac{dk}{2\pi}(k^2 + m^2)^{\frac{1}{2}-s}\frac{\partial\alpha}{\partial k}. \tag{5.17}$$

We see, that all contributions containing $l(\omega)$ are cancelled. Therefore, the regularized boundary energy is given by precisely the same expression as in the commutative case (cf. [16]). Without any further calculations we can read off the renormalized value

$$E_{\rm b}^{\rm ren} = \sqrt{\lambda/2}\,\frac{v_0}{\pi} \tag{5.18}$$

from [16]. Consequently, the renormalized vacuum energy of the kink

$$E_{\rm k}^{\rm ren} = E_{\rm b+k}^{\rm ren} - E_{\rm b}^{\rm ren} = -\sqrt{\lambda/2}\,\frac{v_0}{\pi} \tag{5.19}$$

does not depend on θ and coincides with its' value in the commutative theory.

6. Vacuum expectation value of the new canonical Hamiltonian

There is little doubt in the correctness of the definition of the vacuum energy used in the previous section. However, keeping in mind the applications to saturation of the BPS bound one should also calculate corrections to the new Hamiltonian (3.6) which participates in the supersymmetry algebra.

To calculate vacuum expectation value of the Hamiltonian (3.6) we need the propagators for small fluctuations over the kink background. Let us start in the bosonic sector. Consider eigenfunctions of the operator $P_1(\omega)$,

$$P_1(\omega)\tilde{\phi}_{\omega,\lambda_\omega}(x) = \lambda_\omega^2 \tilde{\phi}_{\omega,\lambda_\omega}(x)\,, \tag{6.1}$$

and normalize them according to the condition

$$\int dx\, \tilde{\phi}_{\omega,\lambda_\omega}^*(x)\tilde{\phi}_{\omega,\lambda_\omega'}(x) = \delta_{\lambda_\omega,\lambda_\omega'}\,. \tag{6.2}$$

We assumed, that there is a boundary in the x-direction, so that the spectrum is discrete. The functions $\tilde{\phi}_{\omega,\lambda_\omega}$ are defined initially on the interval $[-l(\omega), l(\omega)]$ but can be extended to the whole $\mathbb{R}$ as $\tilde{\phi}_{\omega,\lambda_\omega} = 0$ for $|x| > l(\omega)$. The operator $P_1(\omega)$ acts by its analytic formula inside the interval and is extended as multiplication by λ_ω^2 outside the interval and on the boundary. (Of course, as long as $l(\omega)$ is finite the functions $\tilde{\phi}_{\omega,\lambda_\omega}$ cannot be used to expand an arbitrary function on $\mathbb{R}$). The integration in (6.2) can run over $\mathbb{R}$, but the dual formula

$$\sum_\lambda \tilde{\phi}_{\omega,\lambda_\omega}^*(x)\tilde{\phi}_{\omega,\lambda_\omega}(x') = \delta(x - x') \tag{6.3}$$

is valid only if both x and x' belong to $[-l(\omega), l(\omega)]$. Otherwise, the right hand side is zero.

The functions

$$\phi_{\omega,\lambda_\omega}(x^\mu) = e^{-i\omega t}\tilde{\phi}_{\omega,\lambda_\omega}(x) \tag{6.4}$$

are the eigenfunctions of the full kinetic operator acting on fluctuations (restricted to an interval) with eigenvalues $-\omega^2 + \lambda_\omega^2$. The propagator can then be constructed in the standard way as

$$G(x^\mu, x^{\mu\prime}) = \frac{1}{2\pi}\int d\omega \sum_{\lambda_\omega} \frac{\phi_{\omega,\lambda_\omega}(x^\mu)\phi_{\omega,\lambda_\omega}^*(x^{\mu\prime})}{-\omega^2 + \lambda_\omega^2 - i\varepsilon}, \tag{6.5}$$

but the relation $P_1 G(x^\mu, x^{\mu\prime}) = \delta(x^\mu, x^{\mu\prime})$ is true only if both x^1 and $x^{1\prime}$ belong to the intersection of the intervals $[-l(\omega), l(\omega)]$, i.e., to $[-l_0, l_0]$. For $l_0 \to \infty$ one recovers the Feynman propagator. Then,

$$\langle \phi(x^\mu)\phi(y^\nu)\rangle = -iG(y^\nu, x^\mu). \tag{6.6}$$

With the help of this equation one calculates the one-loop vacuum expectation of the bosonic part of the Hamiltonian

$$\langle H\rangle_B = -\frac{i}{2}\int d^2x(-\partial_0^2 - \partial_1^2 + \lambda v_0^2 - \lambda(L(\Phi^2) + R(\Phi^2) + L(\Phi)R(\Phi))_x \times \tag{6.7}$$
$$\times\, G(x^\mu, y^\nu)|_{y^1=x^1,\, y^0=x^0+\sigma}\,.$$

where we introduced a time-splitting regularization with the parameter σ. The operator acting on G should be understood as $-\partial_0^2 + P_1$. The action of P_1 on $\tilde{\phi}_{\omega,\lambda_\omega}$ is already defined above. It is easy to see, that the integrand does not depend on x^0. In order to remove the

corresponding divergence we restrict the integration over x^0 to $[0, T]$ with some finite T. We have,

$$\langle H \rangle_B = -\frac{iT}{2} \int \frac{d\omega}{2\pi} \int dx^1 \sum_{\lambda_\omega} \frac{\omega^2 + \lambda_\omega^2}{-\omega^2 + \lambda_\omega^2 - i\varepsilon} \tilde{\phi}_{\omega,\lambda_\omega}(x^1) \tilde{\phi}_{\omega,\lambda_\omega}^*(x^1) e^{i\omega\sigma}$$

$$= -\frac{iT}{2} \int \frac{d\omega}{2\pi} \sum_{\lambda_\omega} \frac{\omega^2 + \lambda_\omega^2}{-\omega^2 + \lambda_\omega^2 - i\varepsilon} e^{i\omega\sigma} \tag{6.8}$$

Let $\sigma < 0$. The integration contour can be closed in the lower complex half-plane. For each value of ω there is a discrete set of eigenvalues $\{\lambda_\omega^j\}$. Let ω_j be positive solutions of the equation $\omega_j = \lambda_{\omega_j}^j$ (there could be multiple solutions of this equation for each j, but we do not consider such case for simplicity). Then,

$$\langle H \rangle_B = \frac{T}{2} \sum_j \omega_j \left(1 - \frac{d\lambda_\omega^j}{d\omega}\big|_{\omega=\omega_j} \right)^{-1}. \tag{6.9}$$

For $\sigma > 0$ the result is the same.

This formula admits a rather simple interpretation. The factor T appears since our Hamiltonian has the meaning of energy integrated over the time. The expression under the sum is an energy of an excitation with the frequency ω_j. In the commutative limit the derivative in the bracket vanishes, so that each excitation contributes $\frac{1}{2}\omega$. In the NC case, a correction factor appears. The presence of this factor means that the contribution of an individual mode to $\langle H \rangle$ differs from that to E. As we shall see below, due to the supersymmetry this difference does not affect the final result when contributions of all modes, bosonic and fermionic, are taken into account.

For contribution of the fermionic fluctuations one obtains similarly[6]

$$\langle H \rangle_F = -\frac{T}{2} \sum_j \omega_j \left(1 - \frac{d\lambda_\omega^j}{d\omega}\big|_{\omega=\omega_j} \right)^{-1}, \tag{6.10}$$

where, as expected, the overall sign is different from (6.9). ω_j now denote the fermionic frequencies.

Due to the isospectrality of bosonic and fermionic fluctuations on a background of the kink in the presence of boundaries

$$\langle H \rangle_F^{b+k} + \langle H \rangle_B^{b+k} = 0. \tag{6.11}$$

(It is understood that these quantities must be regularized by replacing ω with ω^{1-2s}. The calculations proceed precisely as in the previous section.)

Let us now calculate the boundary contribution $\langle H \rangle^b$ to the vacuum expectation value of the Hamiltonian. An effective free field theory which must be used to calculate boundary

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[6]The only subtlety is the way to extend the eigenfunctions satisfying Robin boundary conditions outside the interval $[-l(\omega), l(\omega)]$. This should be done again by setting these function to zero. Possible discontinuities at the boundary do not play a role. In this way we preserve the isospectrality of P_1 and P_2.

contributions was described in the previous section. The boundary conditions are given by (4.12) and (4.13), and the spectrum of λ_ω^j is defined by the solutions of the equation $f(\omega|\lambda_\omega) = 0$, where

$$f(\omega|\lambda) = \sin(2(k(\lambda)l(\omega) + \alpha(k(\lambda)))), \qquad k(\lambda) = \sqrt{\lambda^2 - m^2}, \tag{6.12}$$

$\alpha(k) = 0$ for Dirichlet conditions, $\alpha(k) = -\arctan(S/k)$ for Robin ones. The quantity

$$h(s) = \sum_j \omega_j^{1-2s} \left(1 - \frac{d\lambda_\omega^j}{d\omega}|_{\omega=\omega_j}\right)^{-1}, \tag{6.13}$$

which is a zeta-regularized expression for the right hand sides of (6.9) and (6.10), can be represented as a contour integral

$$h(s) = \frac{1}{2\pi i} \oint d\omega\, \omega^{1-2s} \left(1 - \frac{d\lambda_\omega}{d\omega}|_{\omega=\lambda_\omega}\right)^{-1} \partial_\omega(\ln f(\omega|\omega)), \tag{6.14}$$

where the contour encircles $[m, \infty[$. One can write

$$\partial_\omega f(\omega|\omega) = [\partial_\omega f(\omega|\lambda) + \partial_\lambda f(\omega|\lambda)]_{\lambda=\omega}. \tag{6.15}$$

On the other hand, the condition $f(\omega|\lambda_\omega) = 0$ defines the dependence of λ_ω on ω. By differentiating this condition, one gets

$$0 = \partial_\omega f(\omega|\lambda_\omega) = [\partial_\omega f(\omega|\lambda)]_{\lambda=\lambda_\omega} + [\partial_\lambda f(\omega|\lambda)]_{\lambda=\lambda_\omega} \frac{d\lambda_\omega}{d\omega}. \tag{6.16}$$

By using (6.15) and (6.16) we rewrite (6.14) as[7]

$$h(s) = \frac{1}{2\pi i} \oint d\omega\, \omega^{1-2s}[\partial_\lambda \ln f(\omega|\lambda)]_{\lambda=\omega}. \tag{6.17}$$

By using the identities which we have just derived one can represent the zeta-regularized boundary contribution to the v.e.v. of H in the form

$$\langle H \rangle^{\mathrm{b}}(s) = \frac{T\mu^{2s}}{4} \frac{1}{2\pi i} \oint d\omega\, \omega^{1-2s}[\partial_\lambda(\ln f_D(\omega|\lambda) - \ln f_R(\omega|\lambda))]_{\lambda=\omega}, \tag{6.18}$$

where $f_{D,R}$ correspond to Dirichlet and Robin boundary conditions respectively (cf. eq. (6.12) and the line below). As in the previous section, on the upper part of the contour we approximate $\sin(2(kl(\omega) + \alpha)$ by $-(1/2i)\exp(-2i(kl(\omega) + \alpha))$, and by $(1/2i)\exp(2i(kl(\omega) + \alpha))$ on the lower part. Then the terms with $l(\omega)$ cancel, and we arrive at the expression

$$\langle H \rangle^{\mathrm{b}}(s) = -T\mu^{2s} \int_m^\infty \frac{d\omega}{2\pi} \omega^{1-2s}[\partial_\lambda \alpha(k(\lambda))]_{\lambda=\omega}. \tag{6.19}$$

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[7]This equation can be also obtained in a different way. As follows from the analysis of [27, 28], the factor $(1 - (d\lambda/d\omega))^{-1}$ is the difference between the spectral density of eigenfrequencies ω_j and the spectral density of the eigenvalues λ for a given ω taken at $\lambda = \omega$. The integral (6.17) is simply a sum over the eigenfrequencies with the latter density.

Next we observe that $[\partial_\lambda \alpha(k(\lambda))]_{\lambda=\omega} = \partial_\omega \alpha(k(\omega))$ with $k(\omega) = \sqrt{\omega^2 - m^2}$ and change the integration variable to k.

$$\langle H \rangle^{\mathrm{b}}(s) = -T \mu^{2s} \int_0^\infty \frac{dk}{2\pi} (k^2 + m^2)^{\frac{1}{2}-s} \partial_k \alpha(k), \qquad (6.20)$$

or,

$$\langle H \rangle^{\mathrm{b}}(s) = T E^{\mathrm{b}}(s). \qquad (6.21)$$

In the heat kernel subtraction scheme $\langle H \rangle^{\mathrm{div}}(s) = T E^{\mathrm{div}}(s)$, so that for the renormalized values we also have the relation

$$\langle H \rangle^{\mathrm{b}}_{\mathrm{ren}} = T E^{\mathrm{b}}_{\mathrm{ren}}. \qquad (6.22)$$

Taking into account (5.10) and (6.11), we conclude that

$$\langle H \rangle^{\mathrm{k}}_{\mathrm{ren}} = T E^{\mathrm{k}}_{\mathrm{ren}} = -T \sqrt{\lambda/2} \frac{v_0}{\pi}. \qquad (6.23)$$

This is a very natural result. It tells us that the interpretation of the new canonical Hamiltonian as the energy integrated over a time interval remains valid also at the one-loop level.

7. Conclusions

In this work we studied quantum corrections to the mass of the kink of supersymmetric NC φ^4. Contrary to the nonsupersymmetric case [9], the counterterm required to remove the divergences is precisely the same as in the commutative theory. The strategy of calculations of the one-loop corrections was taken from [16]. We introduced boundaries, so that the spectrum of the fluctuations becomes discrete. Because of the nonlocality of NC theories, the position of the boundary depends on the frequency of each fluctuation. For the system of the kink and the boundaries, we used the isospectrality of bosonic an fermionic fluctuations which follows from supersymmetry. The total energy of this system vanishes. Then we subtracted the contribution from the boundaries, which was calculated in a relatively simple effective theory. The heat kernel subtraction scheme (which is equivalent to the "no-tadpole" normalization condition in two-dimensional commutative models) gave a value of the mass correction which did not depend on the NC parameter and coincided with the commutative value.

By making use of an unconventional canonical formalism we were able to define supercharges (despite the presence of an infinite number of time derivatives and the absence of locally conserved currents), and to show that the new brackets of these supercharges give an analog of the Hamiltonian and an analog of the central charge. (Note, that the supercharges do generate the supersymmetry transformations, and the Hamiltonian does generate the equations of motion, provided the new canonical brackets are used). This Hamiltonian can be interpreted as the energy integrated over an interval T of the time. The one-loop vacuum expectation value of this Hamiltonian appears to be the quantum correction to the mass of the kink times T, i.e., the picture remains consistent after turning on the quantum effects. Although we have two different definitions of quantum corrections

to the energy (one through a sum over the eigenfrequencies, and the other through the Hamiltonian of the unconventional canonical formalism), both definitions give essentially equivalent results.

In a future publication we are going to calculate quantum corrections to the central charge. This will allow to check whether the quantum BPS bound remains saturated in NC theories. It would also be interesting to consider quantum corrections to solitons in higher dimensional NC theories.

Acknowledgments

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